Group Edge Irregularity Strength of Graphs

Marcin Anholcer

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Notation

- G - simple graph
- $E(G)$ - the edge set of G, $m = |E(G)|$
- $V(G)$ - the vertex set of G, $n = |V(G)|$
- Maximum degree: $\Delta(G)$, minimum degree: $\delta(G)$
- $\mathcal{G}$ - Abelian group, for convenience: 0, $2a$, $-a$, $a-b\dots$

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$s(G)$: Definition

Assign positive integer $w(e) \leq s$ to every edge $e \in E(G)$.

- For every vertex $v \in V(G)$ the *weighted degree* is defined as:

$$wd(v) = \sum_{e \ni v} w(e).$$

- w is irregular if for $v \neq u$ we have $wd(v) \neq wd(u)$.
- ***Irregularity strength*** $s(G)$**:** the lowest s that allows some irregular labeling.
- Introduced by G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, 1988.

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$s(G)$: Some results

- Lower bound:

$$s(G) \geq \max_{1 \leq i \leq \Delta} \frac{n_i + i - 1}{i}$$

- Best upper bound (M. Kalkowski, M. Karoński, F. Pfender, 2009):

$$s(G) \leq \left\lceil \frac{6n}{\delta} \right\rceil$$

- Exact values for some families of graphs (e.g. cycles, grids, some kinds of trees, circulant graphs).

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$$es(G) \geq \max\left\{\frac{m+1}{2}, \Delta(G)\right\}$$

- Best upper bound:

$$es(G) \leq F_n,$$

where F_n is the n^{th} Fibonacci number with seed values $F_1 = 1, F_2 = 2$.

- Exact values for some families of graphs (paths, cycles, stars, double stars, grids).

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- Harmonious graphs (Graham and Sloane, Beals et al., Żak).
- A-cordial labellings (Hovey).
- Edge-magic total labellings (Cavenagh et al.).
- Group distance magic graphs (Froncek).
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$s_g(G)$: Some Results

Theorem (Anholcer, Cichacz, Milanič, 2015)

Let G be arbitrary connected graph of order $n \geq 3$. Then

$$s_g(G) = \begin{cases} n+2 & \text{when } G \cong K_{1,3^{2q+1}-2} \text{ for some integer } q \geq 1 \\ n+1 & \text{when } n \equiv 2 \pmod 4 \wedge G \ncong K_{1,3^{2q+1}-2} \\ n & \text{otherwise} \end{cases}$$

Other results (Anholcer, Cichacz, 2015+): slightly weaker theorem for disconnected graphs, including the results for cyclic groups.

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Lower Bounds

Proposition

For each graph G, $es_g(G) \geq m$.

The above bound is sharp, as e.g. the example of K_5 shows. Computational evidence shows also that even cyclic groups $\mathbb{Z}_{\binom{n}{2}}$ are not enough to label K_n for various $n \geq 6$.

General Upper Bound

Proposition

For each graph G, $es_g(G) \leq p(2F_n)$, where $p(k)$ is the least prime greater than k and F_n is the n^{th} Fibonacci number with seed values $F_1 = 1, F_2 = 2$.

Forests

Proposition

For each forest F, $es_g(F) = m$. Moreover, any weighting of edges is possible for arbitrary choice of labels of one vertex in each component.

Proof.

Given any edge that is still not weighted, if one of the vertices has label a, and the edge is supposed to be weighted with b, it is enough to put $b - a$ on the other vertex. □

Cycles

Theorem

Let C_n be arbitrary cycle of order $n \geq 3$. Then

$$es_g(G) = \begin{cases} n+1 & \text{when } n \equiv 2 \pmod 4 \\ n & \text{otherwise} \end{cases}$$

Moreover respective labeling exists for an arbitrary choice of the label of any vertex.

Remark: in fact, the labeling can be found for any group of order at least $es_g(C_n)$.

Cycles-lower bound

Lemma

If $n \equiv 2 \pmod{4}$*, then* $es_g(C_n) \geq n + 1$.

Cycles-lower bound

Proof.

Assume we can use some $\mathcal{G}$ of order $2(2k+1)$. Obviously $\mathcal{G} = \mathbb{Z}_2 \times \mathcal{G}_1$. There are $2k+1$ elements $(1, a)$ where $a \in \mathcal{G}_1$ and all of them have to appear as the edge weights, so

$$\sum_{e \in E(G)} wd(e) = (1, b_1)$$

For some $b_1 \in \mathcal{G}_1$. On the other hand

$$\sum_{e \in E(G)} wd(e) = 2 \sum_{v \in V(G)} w(v) = (0, b_2)$$

for some $b_2 \in \mathcal{G}_1$. A contradiction. $\square$

Cycles-upper bound

Labeling the vertices distinguishing the edge weights is in this case equivalent to the labeling of the edges distinguishing the vertex weights (we label the line graph, moreover $m=n$). We start with a path and then label remaining edge (or vertex) with 0.

Cycles-upper bound

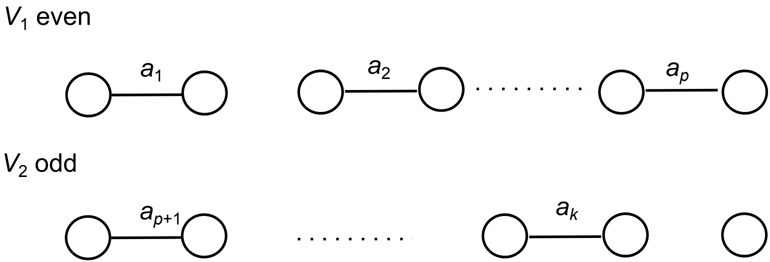
Main idea: alternating paths.

$C(x_i)=C(x_j)$

$C(x_i)\neq C(x_j)$

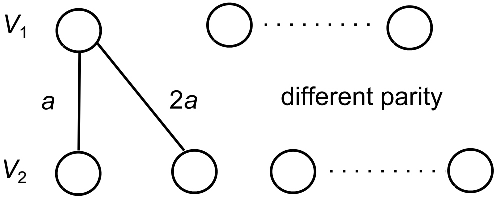
Cycles-upper bound

Case $n = 2k + 1$: take $a_1, \dots, a_k$, $a_i \notin \{a_j, -a_j\}$.



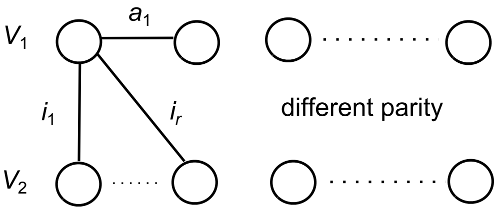
Cycles-upper bound

Case $n = 4k$, one involution - subgroup $\{0, a, 2a, 3a\}$, reduction:



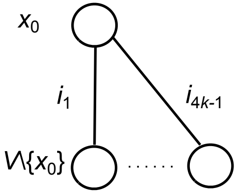
Cycles-upper bound

Case $n = 4k$, $r \leq n/2$ involutions:



Cycles-upper bound

Case $n = 4k$, $r = n - 1$ involutions, $\mathcal{G} = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$



Cycles - upper bound

- Case $n = 4k + 2$, colour classes even: use $\mathcal{G}$ without 0.
- Colour classes odd: we label $K_{3,5}$.

Generalized forests

A **generalized tree** U is a graph constructed in the following way. Given a tree T, we choose some vertices of T and blow each of them to a cycle (former neighbors being now connected to any of the vertices of the cycle). The number of those vertices (cycles) will be denoted by $c(U)$. A union of generalized trees is called **generalized forest**.

Generalized forests

Theorem

Let W be a generalized forest. Then

$$es_g(W) \leq m + \sum_{U \subseteq W} 2c(U) + 1.$$

Generalized forests

Proof - sketch:
Label tree subgraphs and cycle subgraphs separately. In each case we "loose" at most one possible weight (depends on the remainder of the division by 4, and sizes of "linking" trees/paths). The "non-linking" trees do not need additional labels (may be labeled in the end).

Open Problem

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Determine the group edge irregularity strength of arbitrary graph.

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Determine the non-zero group edge irregularity strength of arbitrary graph (neutral element of $\mathcal{G}$ cannot be assigned to any vertex).

Open Problem

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Determine the (non-zero?) group edge irregularity strength of arbitrary planar graph.

Thank You

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